A note on group velocity

By G. B. WHITHAM
Mathematics Department, Massachusetts Institute of Technology

(Received 6 June 1960)

The kinematic approach to group velocity given in Lighthill & Whitham (1955) for one-dimensional waves is extended to cover the general three-dimensional case. The ideas have particular bearing on the theory developed by Ursell (1960) for treating steady wave patterns on non-uniform steady fluid flows.

Although this note was written in ignorance of the fact, all the main ideas presented here are implicit in §§66 and 67 of the book by Landau & Lifshitz (1959). However, these ideas do not seem to be well known to fluid dynamicists, and it was suggested to the author by the editor that a useful purpose would still be served by publishing this note as an expository article amplifying the paragraphs in Landau & Lifshitz. It also serves the original purpose of providing a supplement to Ursell's paper.

1. One-dimensional waves

The following simple approach to group velocity for one-dimensional wave propagation is included as an example of 'kinematic waves' in Lighthill & Whitham (1955, p. 286). Let $k(x,t)$, $\omega(x,t)$ be the local wave number and frequency in the wave train and assume that as the train propagates the number of waves (number of crests, say) is conserved. Balancing the rate of increase $(\partial k/\partial t)\,\delta x$ of the number of waves in a fixed length δx with the net flux $-\delta\omega$, this conservation law is expressed as

$$\frac{\partial k}{\partial t}+\frac{\partial \omega}{\partial x}=0. \tag{1}$$

If ω is a known function of k, usually deduced from assuming a uniform simple harmonic wave train locally, we have

$$\frac{\partial k}{\partial t}+C(k)\frac{\partial k}{\partial x}=0, \tag{2}$$

where
$$C(k)=\frac{d\omega}{dk}. \tag{3}$$

$C(k)$ is the 'group velocity' and equation (2) says that values of k are propagated with the local group velocity $C(k)$ even though the individual crests propagate with the local phase velocity $c=\omega/k$. It should be noted in particular that this result is not limited to a wave packet in which k remains close to some mean value k_0.

The results can be displayed in an (x,t) diagram as shown in figure 1. The characteristic lines $dx/dt=C(k)$ are shown non-overlapping. This is the situation, for example, in the gravity waves problem of the release of an initial elevation of the water surface, *after a sufficient time has elapsed*. Then the longer waves with larger group velocity are at the front of the wave; the leading characteristic would correspond to the maximum group velocity $(g \times \text{depth})^{\frac{1}{2}}$ and would play the role of a wave front. For earlier times one can think of extending the characteristics in figure 1 backwards in time until they do overlap. Such a region of overlapping characteristics could be interpreted as the region in which the initial elevation, perhaps consisting of a single crest, is breaking down into a whole series of crests, and in this period the conservation equation (1) does not apply. This corresponds to the fact that in the exact solution of the problem by means of Fourier integrals, the simple description of the motion involving the group velocity is obtained only in the asymptotic behaviour limited to large t (see Jeffreys & Jeffreys 1956, §17·08).

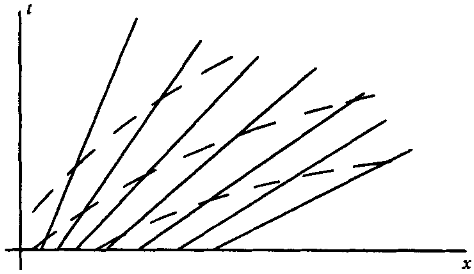

FIGURE 1. The full lines represent the characteristics $dx/dt = C(k)$ and k is constant on each one. The broken lines represent the paths of individual wave crests $dx/dt = c(k)$ in the case $c > C$.

However, a different kind of overlapping may occur where two distinct sets of waves are superposed; an example is when short capillary waves are superposed on gravity waves. Then the description by (1) applies to each set. A similar type of overlapping occurs in two and three dimensions when two wave systems cross at an angle; an example occurs in ship waves. In either type it seems to be unnecessary to rule out such overlapping solutions and introduce shocks as is necessary for the analogous simple waves of gas dynamics and the other examples of kinematic waves given in the above reference.

It should be remarked that when $\omega(k)$ is determined by discussing the local propagation, the position x may be involved as a parameter so that $\omega = W(k, x)$. Then, in (1) we have

$$\frac{\partial k}{\partial t}+\frac{\partial W}{\partial k}\frac{\partial k}{\partial x}+\frac{\partial W}{\partial x} = 0 \quad \text{and} \quad \frac{\partial \omega}{\partial t}+\frac{\partial W}{\partial k}\frac{\partial \omega}{\partial x} = 0. \tag{4}$$

Therefore, ω is constant along the characteristic curves $dx/dt = \partial W/\partial k = C$, but k is not.

2. Two and three dimensions

The clearest derivation of the extension is to observe that (1) expresses the existence of a function $\phi(x, t)$ and a set of curves in the (x, t) plane given by $\phi(x, t) =$ constant which can be recognized as waves. These curves could easily be defined in terms of the motion of 'crests' and 'troughs' provided none of these disappear. Then (1) follows with $k = \partial\phi/\partial x$ and $\omega = -\partial\phi/\partial t$, which are of course the correct quantities for wave-number and frequency in terms of ϕ. The function ϕ is the 'phase function'.

Now suppose that a set of wave surfaces $\phi(x,t) = $ constant can be recognized as waves in two or three dimensions. Then, defining the vector wave number $\mathbf{k}$ as $\nabla\phi$ and the frequency ω as $-\partial\phi/\partial t$, we have

$$\frac{\partial \mathbf{k}}{\partial t} + \nabla\omega = 0. \tag{5}$$

The phase velocity $\mathbf{c}$ is the velocity of the surface $\phi(\mathbf{x},t) = $ constant; it is in the normal direction, i.e. parallel to $\mathbf{k}$, and its magnitude is $-\phi_t/|\nabla\phi| = \omega/k$. The existence of a scalar field ϕ corresponds to the assumption of wave conservation. Since $\mathbf{k} = \nabla\phi$, it follows that

$$\operatorname{curl}\mathbf{k} = 0. \tag{6}$$

The irrotationality of $\mathbf{k}$ could also be the starting point in deducing the existence of ϕ. For, introducing $\mathbf{k}$ as the vector wave-number, conservation of waves requires the total number (with correct sign) crossing any closed curve be zero; hence

$$\oint \mathbf{k}.d\mathbf{s} = 0$$

and (6) follows.

Again it is assumed that the dependence of ω on $\mathbf{k}$, and possibly on location $\mathbf{x}$, has been deduced by local arguments so that

$$\omega = W(\mathbf{k},\mathbf{x}) \tag{7}$$

is known. Then, given appropriate initial or boundary conditions for $\mathbf{k}$, equations (5) and (7) determine the vector field $\mathbf{k}$ from which the flow pattern given by ϕ can be calculated. Substituting for ω in (6), we have three equations for the components k_i of $\mathbf{k}$

$$\frac{\partial k_i}{\partial t} + C_j \frac{\partial k_j}{\partial x_i} + F_i = 0, \tag{8}$$

where

$$C_j = \frac{\partial W}{\partial k_j}, \quad F_i = \frac{\partial W}{\partial x_i}.$$

The velocity $C_j(\mathbf{k},\mathbf{x}) = \partial W/\partial k_j$ is the three-dimensional group velocity. The interpretation of (8) is immediate when we note that $\mathbf{k}$ is irrotational so that $\partial k_j/\partial x_i = \partial k_i/\partial x_j$ and (8) can be rewritten

$$\frac{\partial k_i}{\partial t} + C_j \frac{\partial k_i}{\partial x_j} = -F_i. \tag{9}$$

The left-hand side is the rate of change of k_i following a point moving with the group velocity $\mathbf{C}$. In principle, at least, this determines $\mathbf{k}$ at all future times and ϕ is calculated from $\nabla\phi = \mathbf{k}$.

When the frequency $\omega = W(\mathbf{k},\mathbf{x})$ is independent of $\mathbf{x}$, $F_i = 0$ and (9) states that values of k_i propagate unchanged in magnitude with the group velocity. But, in this case, $\mathbf{C}$ is a function of $\mathbf{k}$ only; therefore, the propagation is with constant velocity along straight lines. In general the propagation velocity is different on the different lines, and the lines are not parallel. This is the generalization of the one-dimensional result represented in figure 1. In the $(\mathbf{x}, t)$ space the characteristic curves $d\mathbf{x}/dt = \mathbf{C}$ are straight lines. In the special case of a wave packet with $\mathbf{k}$ close to $\mathbf{k}_0$ everywhere, the lines are parallel if we take the approximation $\mathbf{C}(\mathbf{k}) = \mathbf{C}(\mathbf{k}_0)$; this would be the 'linear theory'. When W involves position $\mathbf{x}$, $\mathbf{k}$ is no longer constant along the characteristic curves. However, if the time does not appear explicitly in W, the frequency ω is constant on characteristics. For, taking the scalar product of (5) with $\mathbf{C}$, we have

$$\frac{\partial \omega}{\partial t} + C_i \frac{\partial \omega}{\partial x_i} = 0, \tag{10}$$

since $C_i (\partial k_i / \partial t) = \partial \omega / \partial t$.

Of course, one could equally well work with the original phase function $\phi(\mathbf{x}, t)$ and write (7) as

$$\frac{\partial \phi}{\partial t} + W\left(\frac{\partial \phi}{\partial x_i}, x_i\right) = 0. \tag{11}$$

But the standard methods of dealing with such equations introduce $\partial\phi/\partial t$ and $\partial\phi/\partial x_i$ as new variables and use the characteristic forms (9) and (10). Thus, there is no essential difference. Equation (11) has the same form as the Hamilton–Jacobi equation and the previous equation (9) is equivalent to

$$\frac{dk_i}{dt} = -\frac{\partial W}{\partial x_i}, \quad \frac{dx_i}{dt} = \frac{\partial W}{\partial k_i},$$

which are the corresponding Hamilton equations. The relation of wave motion to Hamilton's equations is familiar in wave mechanics, but this approach has not been widely used to introduce the general treatment (not limited to a wave packet) of group velocity for classical waves.

One important point is that the direction of the group velocity $\mathbf{C}$ will be the same as the phase velocity $\mathbf{c}$ (which in turn is parallel to $\mathbf{k}$) if and only if ω depends on the magnitude of $\mathbf{k}$ only and not on the direction of $\mathbf{k}$. The first part is trivial: if $\omega = f(k)$ where $k = |\mathbf{k}|$, then $\partial\omega/\partial k_i = f'(k)\, k^{-1} k_i$ and the result follows. For the second part, assume that

$$\frac{\partial \omega}{\partial k_i} = g(k_1, k_2, k_3)\, k_i. \tag{12}$$

Then, for $i \neq j$,
$$\frac{\partial^2 \omega}{\partial k_i \partial k_j} = \frac{\partial g}{\partial k_i} k_j = \frac{\partial g}{\partial k_j} k_i;$$

hence
$$\frac{\partial g}{\partial (k_i^2)} = \frac{\partial g}{\partial (k_j^2)} \tag{13}$$

for all i, j with $i \neq j$, and it follows that g is a function of k^2 only. Then, integrating (12), we have that ω is a function of k.

3. Surface wave pattern on a non-uniform steady flow

The theory developed by Ursell (1960) can be considered as a special case of the above ideas. In a steady flow pattern the phase function ϕ is independent of t and the frequency ω is zero. If the known flow velocity is $\mathbf{u}(\mathbf{x})$ and if the phase velocity for local propagation relative to fluid at rest would be $\mathbf{c}_0$, then the resultant phase velocity $\mathbf{c}$ is the sum of $\mathbf{c}_0$ and the component of $\mathbf{u}$ in the $\mathbf{k}$ direction, i.e.

$$\mathbf{c} = (\mathbf{u}.\hat{\mathbf{k}})\hat{\mathbf{k}} + \mathbf{c}_0, \quad \text{where} \quad \mathbf{k} = k\hat{\mathbf{k}}.$$

Hence, the frequency is

$$\omega = \mathbf{c}.\mathbf{k} = (\mathbf{u}+\mathbf{c}_0).\mathbf{k} = \mathbf{u}.\mathbf{k}+\omega_0.$$

For a steady pattern, therefore,

$$(\mathbf{u}+\mathbf{c}_0).\mathbf{k} = 0. \tag{14}$$

Since $\mathbf{u}(\mathbf{x})$ and $\mathbf{c}_0(\mathbf{k},\mathbf{x})$ are assumed to be known, this is a functional relation between k_1 and k_2. It is noted directly in Ursell's paper as the condition that the wave crests remain steady in the flow.

To this relation we simply add the irrotationality condition

$$\operatorname{curl}\mathbf{k} = 0, \tag{15}$$

and (14), (15) are the equations to determine the two components (k_1, k_2) of the wave-number $\mathbf{k}$. As Ursell finds, the characteristics of these equations are lines in the direction of the resultant of the stream velocity $\mathbf{u}$ and the group velocity $\partial\omega_0/\partial k_i$. This result clearly fits in with the more general treatment given here. Notice that in the time dependent problem, (15) plays a subsidiary role as an initial condition, since it then follows from (5) that curl $\mathbf{k}$ remains zero. However, for the steady wave pattern (15) becomes the basic conservation equation.

The 'ship wave' pattern, produced by a fixed disturbance in the stream, has two superimposed sets of waves in the wedge-shaped region behind the obstacle. This is an example of acceptable overlapping noted in §1.

4. Variation of amplitude and energy propagation

For a uniform medium in which component simple harmonic waves (constituting the full disturbance) propagate without change in amplitude, energy propagates with the group velocity. Here, energy means the integral of the square of the amplitude over a region of space, and propagation with the group velocity means that if we consider the volume $V(t)$ enclosing a given set of points each moving with the appropriate group velocity, the energy in V does not change with time. A full derivation and discussion for one-dimensional waves is given in Jeffreys & Jeffreys (1956). The three-dimensional case goes through in complete analogy. The volume should contain several waves and in a given problem this result applies after a sufficiently large time; the time should be large enough for the Fourier integral over the harmonic components to be approximated by the first term in its asymptotic expansion for large t. This is also the requirement for the kinematic properties of group velocity to apply.

Thus we have the result

$$\text{amplitude} \propto \left(\frac{\Delta V_0}{\Delta V}\right)^{\frac{1}{2}}, \tag{16}$$

where ΔV is a small volume which still contains several waves, ΔV_0 is its initial value and the amplitude is an average one for the waves in ΔV. For a uniform medium, the group velocity remains constant on lines of propagation and we have

$$\frac{\Delta V}{\Delta V_0} \propto \det\left|\frac{\partial C_i}{\partial k_j}\right| t = \det\left|\frac{\partial^2 W}{\partial k_i \partial k_j}\right| t, \tag{17}$$

provided the distance propagated is much greater than the diameter of ΔV. Expressions (16) and (17) give the typical amplitude factor which appears after applying the method of stationary phase to the Fourier integral.

For a non-uniform medium we expect the energy to propagate with the group velocity in the same way, provided that typical length scales in the variation of the medium are large compared with typical wave lengths. This provides a simple approximate method for calculating amplitudes. Here energy means the physical energy which is related to the squares of velocity amplitude or pressure amplitude by factors which now depend on position.

This research was supported by the United States Air Force under Contract AF49(638)-708, monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

REFERENCES

JEFFREYS, H. & JEFFREYS, B. S. 1956 *Methods of Mathematical Physics* (3rd ed.). Cambridge University Press.

LANDAU, L. D. & LIFSHITZ, E. M. 1959 *Fluid Mechanics*. London: Pergamon Press.

LIGHTHILL, M. J. & WHITHAM, G. B. 1955 *Proc. Roy. Soc.* A, **229**, 281–316.

URSELL, F. 1960 *J. Fluid Mech.* **9**, 333–46.